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Finite Concrete Logics: Their Structure and Measures on Them

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After making a brief survey of results on finite concrete logics obtained during the last 16 years, we present new ones due to the authors. We especially concentrate on developing the duality theory and handling problems of extending measures and signed measures.

1. INTRODUCTION

Orthomodular posets (OMPs) (Kalmbach, 1983) and a number of more general structures (orthoalgebras, D-posets, etc.) were introduced to generalize the logical approach to the foundations of quantum mechanics due to G. Birkhoff and J. von Neumann. Sub-OMPs of the Boolean algebra of all subsets of a set are known as concrete logics. As all OMPs can, the latter can serve as domains for measures and signed measures, which has resulted in the creation of the so-called generalized measure theory (Gudder, 1979, 1984) generalizing the classical one for σ-algebras. Finite concrete logics provide new subjects for combinatorial and measure-theoretic investigation.

Examples of finite concrete logics have surely been of interest to almost everyone who has ever dealt with orthomodular structures. Still, finite concrete logics have remained a little-studied area, as many investigators have not gone further than considering counterexamples.

Here, we describe some general results on finite concrete logics obtained since 1980 (in particular, by the authors). In the subsequent sections, we contribute to developing the general theory. The proofs are omitted and will appear elsewhere (e.g., Ovchinnikov, 1996).

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Let Ω be a set and $\mathcal{P}(\Omega)$ the Boolean algebra of all subsets of Ω. A *concrete logic* (c.l.) (see, e.g., Sherstnev, 1968; Gudder, 1979) on Ω is a subset $\mathcal{E}$ of $\mathcal{P}(\Omega)$ satisfying (1) $\Omega \in \mathcal{E}$; (2) $X \in \mathcal{E} \Rightarrow \Omega \setminus X \in \mathcal{E}$; (3) $X, Y \in \mathcal{E}, X \cap Y = \emptyset \Rightarrow X \cup Y \in \mathcal{E}$.

A finitely orthoadditive mapping from $\mathcal{E}$ to $[0, +\infty)$ (resp., $\mathbf{R}$) is called a *measure* [resp., a *signed measure* (s.m.)] on $\mathcal{E}$. A measure μ on $\mathcal{E}$ is called a *state* in case $\mu(\Omega) = 1$. A state μ on $\mathcal{E}$ is called *pure* if μ is an extreme point of the convex set of all states on $\mathcal{E}$ and *two-valued* if $\mu(\mathcal{E}) = \{0, 1\}$. Let $\omega \in \Omega$. Define the two-valued state μ_w on $\mathcal{E}$ as

$$\mu_\omega(X) = \begin{cases} 0 & \text{if} \quad \omega \notin X \\ 1 & \text{if} \quad \omega \in X \end{cases} \qquad (X \in \mathcal{E})$$

The μ_w is called the *point state* on $\mathcal{E}$ defined by ω.

Let $n, m \in \mathbf{N}$. Define a c.l. $\mathcal{L}_m^n$ (Prather, 1980) on $\{1, \ldots, nm\}$ by

$$\mathcal{L}_m^n = \{X \subset \{1, \ldots, nm\} | \text{card} X \equiv 0 \ (\text{mod} m)\}$$

Next, suppose that $n \geq 3$. Prather (1980) showed that the C_{nm}^m m-element subsets of $\{1, \ldots, nm\}$ are generated by complementation of disjoint unions from the $nm - 1$ subsets $\{1, \ldots, m\}, \ldots, \{nm\text{-}m, \ldots, nm - 1\}, \{nm - m + 1, \ldots, nm - 1, 1\}, \ldots, \{nm - 1, 1, \ldots, m - 1\}$. This in particular means that every s.m. on $\mathcal{L}_m^n$ uniquely extends to an s.m. on $\mathcal{P}(\{1, \ldots, nm\})$. Another proof of the latter result was obtained by Sultanbekov (1992a), who also showed that every pure state on $\mathcal{L}_m^n$ is either point or of the form $F_\lambda | \mathcal{L}_m^n$ for some $\lambda \in \{1, \ldots, nm\}$, where F_λ is an s.m. on $\mathcal{P}(\{1, \ldots, nm\})$ defined as

$$F_\lambda(\{\omega\}) = \begin{cases} \dfrac{1}{m(n-1)} & \text{if} \quad \omega \neq \lambda \\ -\dfrac{m-1}{m(n-1)} & \text{if} \quad \omega = \lambda \end{cases} \qquad (\omega \in \{1, \ldots, nm\})$$

In particular, every two-valued state on $\mathcal{L}_m^n$ is point. Also (Sultanbekov, 1992a), if μ is a state on $\mathcal{L}_m^n$ taking exactly two different values on the m-element subsets of $\{1, \ldots, nm\}$, then there exist unique $\lambda \in \{1, \ldots, nm\}$ and $t \in [0, 1/n) \cup (1/n, 1/(n - 1)]$ such that $\mu = \nu_{\lambda,t} | \mathcal{L}_m^n$, where $\nu_{\lambda,t}$ is an s.m. on $\mathcal{P}(\{1, \ldots, nm\})$ given by

$$\nu_{\lambda,t}(\{\omega\}) = \begin{cases} 1 - t\left(n - \dfrac{1}{m}\right) & \text{if} \quad \omega = \lambda, \\ \dfrac{t}{m} & \text{if} \quad \omega \neq \lambda \end{cases} \qquad (\omega \in \{1, \ldots, nm\})$$

Ovchinnikov (1985) (see also Sultanbekov, 1992a) proved that for every authomorphism u of $(\mathcal{L}_m^n, \subset)$ as a poset there exists a unique permutation $\mathbf{ø}$ of $\{1, \ldots, nm\}$ with $u(X) = \mathbf{ø}(X)$ $(X \in \mathcal{L}_m^n)$.

Let $n, L \in \mathbf{N}$, $n \geq 2$, and $L \geq 2$. Put $N = nL$ and $\Omega = \{0, \ldots, N-1\}$. Note that Ω is an abelian group with respect to the addition modulo N. Let Σ be the least, with respect to inclusion, c.l. on Ω containing all sets of the form $k + \{0, \ldots, L-1\}$, where $k \in \Omega$ (Gudder and Marchand, 1980). Gudder and Marchand (1980) (see also Ovchinnikov, 1992) proved that every s.m. on Σ extends to an s.m. on $\mathcal{P}(\Omega)$. Ovchinnikov (1992) showed that if $n \geq 3$ or $n = L = 2$, then every measure on Σ extends to a measure on $\mathcal{P}(\Omega)$, and if $n = 2, L \geq 3$, then there exists a measure on Σ which does not extend to a measure on $\mathcal{P}(\Omega)$.

Let Ω be a finite set and $\mathcal{E}$ a c.l. on Ω. $\mathcal{E}$ is called *regular* if every s.m. on $\mathcal{E}$ extends to an s.m. on $\mathcal{P}(\Omega)$ and *positive* if every measure on $\mathcal{E}$ which extends to an s.m. on $\mathcal{P}(\omega)$ also extends to a measure on $\mathcal{P}(\Omega)$ (Ovchinnikov, 1994a). Let $V(\mathcal{E})$ denote the real vector space of all s.m.'s on $\mathcal{E}$.

Now, let us give a brief account of the duality theory of finite c.l.'s as in Ovchinnikov (1994a). Put $\mathcal{M} = \{\mu \in V(\mathcal{P}(\Omega)) | \mu(\Omega) = 0\}$. For arbitrary $S \subset \mathcal{M}$ and $T \subset \mathcal{P}(\Omega)$ put

$$S^\circ = \{X \in \mathcal{P}(\Omega) | \forall \mu \in S(\mu(X) = 0)\}$$

$$T^\circ = \{\mu \in \mathcal{M} | \forall X \in T(\mu(X) = 0)\}$$

Obviously S° is a c.l. on Ω, and T° is a linESCear subspace of $\mathcal{M}$. The couple of mappings $^\circ$: $\mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{P}(\Omega))$ and $^\circ$: $\mathcal{P}(\mathcal{P}(\Omega)) \to \mathcal{P}(\mathcal{M})$, $\mathcal{P}(\mathcal{M})$, and $\mathcal{P}(\mathcal{P}(\Omega))$ being ordered by the inclusion is a Galois correspondence. $S \in \mathcal{P}(\mathcal{M})(T \in \mathcal{P}(\mathcal{P}(\Omega)))$ is called *closed* if $S = S = S^{\circ\circ}$ (resp., $T = T^{\circ\circ}$). If $\mathcal{E}$ is closed, then there exists $\mu \in \mathcal{E}^\circ$ (called *polarifier* for $\mathcal{E}$) with $\mathcal{E} = \{\mu\}^\circ$. The following three conditions are equivalent:

(i) $\mathcal{E}$ is regular.
(ii) $\dim\mathcal{E}^\circ + \dim V(\mathcal{E}) \leq \mathrm{card}\Omega$.
(iii) $\dim\mathcal{E}^\circ + \dim V(\mathcal{E}) = \mathrm{card}\Omega$.

If $\mathcal{E}$ is positive, then $\mathcal{E}$ is closed. Even for $\mathrm{card}\Omega = 6$, there exist a closed nonpositive, a positive nonregular, and a regular nonclosed c.l. on Ω.

Sultanbekov (1992b, 1993) examined representations as c.l.'s for OMPs whose Greechie diagrams are n-polygons with three atoms on each side (see also Ovchinnikov, 1985, 1994b). Sultanbekov (1995) introduced a notion of a "best" extension for an s.m. on a finite c.l. The well-known theorem by G. Birkhoff on doubly stochastic matrices may be viewed as a result on measures on a suitable finite c.l. (Ovchinnikov 1985, 1994a).

2. THE DUALITY

Let Ω be a finite set again, and $\mathscr{E}$ a c.l. on Ω.

Definition 2.1. A set $X \subset \Omega$ is called $\mathscr{E}$-*strange* if $Y \not\subset X$ for every $Y \in \mathscr{E} \setminus \{\emptyset\}$. Let $\mathrm{St}(\mathscr{E})$ denote the set of all $\mathscr{E}$-strange subsets of Ω.

Let $\alpha(\mathscr{E})$ stand for the set of all *atoms* in $\mathscr{E}$ (i.e., minimal, with respect to inclusion, elements of $\mathscr{E} \setminus \{\emptyset\}$). Every element of $\alpha(\mathscr{E}^{\circ\circ}) \setminus \mathscr{E}$ as well as every set of the form $X \setminus \{\omega\}$, where $X \in \alpha(\mathscr{E})$, $\omega \in X$, can serve as an example of an $\mathscr{E}$-strange set.

Definition 2.2. $\mathscr{E}$ is called *locally positive* if for every $X \in \mathrm{St}(\mathscr{E})$ there exists $\mu \in \mathscr{E}^{\circ}$ satisfying $\mu(\{\omega\}) > 0$ for all $\omega \in X$.

Obviously if $\mathscr{E}$ is locally positive, then $\mathscr{E}$ is closed.

Theorem 2.3. If $\mathscr{E}$ is positive, then $\mathscr{E}$ is locally positive.

Let $\Lambda \subset \mathbf{R}^2$ be finite. Denote by $\Delta(\Lambda)$ the least, with respect to inclusion, c.l. on Λ containing all sets of the form $\pi_1^{-1}(A)$, $\pi_2^{-1}(A)$, or $(\pi_1 + \pi_2)^{-1}(A)$, wherein $\pi_1(x, y) = x$, $\pi_2(x, y) = y$ $((x, y) \in \Lambda)$, and $A \subset \mathbf{R}$ is Borel. It is easy to verify that $\Delta(\{0, \ldots, n-1\} \times \{0, \ldots, m-1\})$ is not locally positive whenever $n \geq 4$ and $m \geq 4$. By Theorem 2.3, the c.l. is not positive. This negatively solves the long-standing and problem discussed during the Second Winter School on Measure Theory at Liptovsky Jan of whether every $\Delta(\Lambda)$ is positive.

Definition 2.4. Let $X \subset \Omega$. The *polar rank* of X with respect to $\mathscr{E}$, $\mathrm{pr}_{\mathscr{E}}(X)$, is defined as $\mathrm{pr}_{\mathscr{E}}(X) = \dim V_X$, where $V_X = \{\mu|\mathscr{P}(X)|\mu \in \mathscr{E}^{\circ}\}$.

Definition 2.5. $\mathscr{E}$ is called *filled* if $\mathrm{pr}_{\mathscr{E}}(X) = (\mathrm{card} X) - 1$ for every $X \in \alpha(\mathscr{E})$.

Theorem 2.6. $\mathscr{E}$ is filled if and only if $\forall X \in \alpha(\mathscr{E})\ \forall \omega \in X\ \exists \mu \in \mathscr{E}^{\circ} \forall \tau \in X \setminus \{\omega\}(\mu(\{\tau\}) > 0)$.

Corollary 2.7. If $\mathscr{E}$ is locally positive, then $\mathscr{E}$ is filled.

Corollary 2.8. If $\mathscr{E}$ is locally positive, then $\mathrm{card} X \leq (\dim \mathscr{E}^{\circ}) + 1$ for all $X \in \alpha(\mathscr{E})$.

Note that $\Delta(\{0, 1, 2, 3\}^2)$ is closed and filled, though not locally positive. The closedness and the filledness of $\mathscr{E}$ do not entail each other even in the case $\mathrm{card}\Omega = 6$.

Theorem 2.9. $\Delta(\{0, \ldots, n-1\} \times \{0, \ldots, m-1\})$ is regular for all $n, m \in \mathbf{N}$.

The proof (which we omit) attracts the Zerbe–Gudder theorem on the additivity of integrals (Zerbe and Gudder, 1985).

Remark 2.10. It follows from Navara and Ptak (1983) that every two-valued state on $\Delta(\Lambda)$ is point for every finite $\Lambda \subset \mathbf{R}^2$.

3. THE ABSTRACT CLOSEDNESS

As in Section 2, let Ω be a finite set, and $\mathscr{E}$ a c.l. on Ω.

Definition 3.1. Let G be an abelian group. $\mathscr{E}$ is called *G-abstractly closed* if there exists a mapping F: $\Omega \to G$ with $\mathscr{E} = \{X \subset \Omega | \Sigma_{\omega \in X} F(\omega) = 0\}$.

Note that $\mathscr{E}$ is closed if and only if $\mathscr{E}$ is $(\mathbf{R}, +)$-abstractly closed. Next, $\mathscr{L}_m^n$ is $(\mathbf{Z}_m, +)$-abstractly closed for all $n, m \in \mathbf{N}$, but is never closed provided that $n \geq 2$ and $m \geq 2$. If G is an abelian group, then $\mathscr{E}$ is G-abstractly closed just in case $\mathscr{E}$ is the kernel (Navara, 1993, n.d.; Mayet and Navara, 1995) of a G-valued measure μ on $\mathscr{P}(\Omega)$ with $\mu(\Omega) = 0$. There exist c.l.'s on finite sets which are not G-abstractly closed for any Abelian group G (Navara, n.d.; see also Ovchinnikov, 1996).

Definition 3.2. $\mathscr{E}$ is called *symmetric* if $X, Y \in \mathscr{E} \Rightarrow X + Y \in \mathscr{E}$, $+$ being the symmetric difference.

Denote by $\mathbf{Z}_2^\infty$ the direct sum of countably many copies of $(\mathbf{Z}_2, +)$.

Now, we recall the rules of the game Nim. Two players participate. There are several heaps, and each heap contains several things. By a move, a player chooses a heap and takes away an arbitrary number of things from it only, at least one and perhaps all. The players alternate their moves. The player who makes the last move wins.

A *position*, in Nim, is the corresponding finite set of heaps. Let A and B be the players, and let A begin. Let Π be a fixed position in which B possesses a winning strategy. Denote by $\mathscr{L}(\Pi)$ the set of all $X \subset \Pi$ such that B has a winning strategy in the position X. Let us show that $\mathscr{L}(\Pi)$ is a c.l. on Π. We will do this without making use of the generally well-known description of the winning strategies in Nim.

(1) By definition, $\Pi \in \mathscr{L}(\Pi)$. (2) Let $X \in \mathscr{L}(\Pi)$. Suppose that $\Pi \backslash X \notin \mathscr{L}(\Pi)$. Then A has a winning strategy in $\Pi \backslash X$. Let us show that A has a winning strategy in Π, and this will be a contradiction. Let A begin in accordance with his winning strategy in $\Pi \backslash X$ and separately play within X or $\Pi \backslash X$ using winning strategies for B or A, respectively. It is clear that A will win. (3) Let $X, Y \in \mathscr{L}(\Pi)$ satisfy $X \cap Y = \emptyset$. Then $X \cup Y \in \mathscr{L}(\Pi)$, as B can separately play within X or Y according to winning strategies for B.

Recall that two concrete logics, $\mathscr{E}_1$ on a set Ω_1 and $\mathscr{E}_2'$ on a set Ω_2, are referred to as *isomorphic* ones if there exists a bijection $\phi : \Omega_1 \to \Omega_2$ satisfying $X \in \mathscr{E}_1 \Leftrightarrow \phi(X) \in \mathscr{E}_2$ for every $X \subset \Omega_1$.

Theorem 3.3. The following three conditions are equivalent:

1. $\mathscr{E}$ is symmetric.
2. $\mathscr{E}$ is $\mathbf{Z}_2^\infty$-abstractly closed.
3. There exists a position Π in Nim such that B has a winning strategy in Π and $\mathscr{E}$ is isomorphic to $\mathscr{L}(\Pi)$.

4. APPLYING A COMPUTER

We refer to Sultanbekov (1996), where the results of a computer treatment of finite c.l.'s are presented.

5. OPEN QUESTIONS

5.1. Is every locally positive c.l. on a finite set positive?

5.2. Is every $\Delta(\Lambda)$, $\Lambda \subset \mathbf{R}^2$ being finite, (a) regular? (b) closed? (c) filled?

5.3. Is every c.l. on a finite set isomorphic, as an OMP, to a c.l. on a finite set which is G-abstractly closed for some abelian group G? [This is a version of a problem posed by Navara (n.d.).]

5.4. Which c.l.'s on finite sets can be represented with combinatorial games similarly to symmetric ones?

5.5. Give a direct "playing" proof of that $\mathscr{L}(\Pi)$ is symmetric.

5.6. Can every s.m. on a c.l. on a finite set be represented as a linear combination of two-valued states on the c.l.?

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